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The Kowalevskaya exponents and rational integrability of polynomial differential systems

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Abstract

This paper primarily grows from the paper of Llibre and Zhang [J. Llibre, X. Zhang, Polynomial first integrals for quasi-homogeneous polynomial differential systems, Nonlinearity 15 (2002) 1269–1280] with the following essential generalizations: (i) we prove that the link established in the mentioned paper between the Kowalevskaya exponents and the degree of the polynomial first integrals holds not only for $(1, \dots, 1)$-2 type systems but also for any $(s_1, \dots, s_n)$-d type systems. (ii) by using different methods, we obtain necessary and sufficient conditions for planar (s_1, s_2)-d systems to have rational first integrals, whereas in the mentioned paper, only (s_1, s_2)-2 type systems and only polynomial integrability are considered.

As an application of the methods and the results, we present an illustrative and well studied example to show its non-existence of polynomial first integrals.

1. Introduction and the main results

In this paper we shall study polynomial and rational integrability of quasi-homogeneous polynomial differential systems. A polynomial differential system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{P}(\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{C}^n, \tag{1}$$

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where $\boldsymbol{P}(\boldsymbol{x}) = (P_1(\boldsymbol{x}), \dots, P_n(\boldsymbol{x}))$, $P_i \in \mathbf{C}[\boldsymbol{x}]$, $i = 1, \dots, n$, is called $\boldsymbol{s} = (s_1, \dots, s_n)$-*quasi-homogeneous of weight degree* d, or $\boldsymbol{s}$-d type for simplicity, if $P_i(\boldsymbol{x})$ is an $\boldsymbol{s}$-$(s_i - 1 + d)$ type polynomial, $i = 1, \dots, n$.

Here, conventionally, a polynomial $H(\boldsymbol{x})$ is said to be $\boldsymbol{s}$-d type quasi-homogeneous if the relation $H(l^{s_1}x_1, \dots, l^{s_n}x_n) = l^d H(x_1, \dots, x_n)$ holds for all $l \in \mathbf{R}$.

Since if $(s_1, \dots, s_n)$ has a common factor k for some integer $k > 1$, then $k|(d-1)$ and any $\boldsymbol{s}$-d type system is also $\frac{\boldsymbol{s}}{k}$-$(\frac{\mathrm{d}-1}{k} + 1)$ type, therefore without loss of generality, throughout the paper, we shall always assume that $(s_1, \dots, s_n) = 1$. It is clear that $(1, \dots, 1)$-d type system coincides with the classical homogeneous polynomial system of degree d.

Note that if system (1) is $\boldsymbol{s}$-d type with $d > 1$, then it is invariant under the change $x_i \to l^{w_i} x_i$, $t \to l^{-1}t$, where $w_i = s_i/(d-1)$. Naturally, we introduce notation $\boldsymbol{w} := \boldsymbol{s}/(d-1)$ and call a point $\boldsymbol{c}^0 = (c_1^0, \dots, c_n^0) \in \mathbb{C}^n \backslash \{0\}$ a *balance* of (1) if it satisfies the algebraic equation $\boldsymbol{P}(\boldsymbol{c}) + \boldsymbol{wc} = \mathbf{0}$, where $\boldsymbol{wc} := (w_1c_1, \dots, w_nc_n)$.

To each balance $\boldsymbol{c}$, we introduce a matrix

$$K(\boldsymbol{c}) = D\boldsymbol{P}(\boldsymbol{c}) + \mathrm{diag}(w_1, \dots, w_n),$$

where $D\boldsymbol{P}(\boldsymbol{c})$ is the differential of $\boldsymbol{P}$ evaluated at $\boldsymbol{c}$ and $\mathrm{diag}(w_1, \dots, w_n)$ is the diagonal matrix with diagonal elements $(w_1, \dots, w_n)$. The eigenvalues of $K(\boldsymbol{c})$ are called the *Kowalevskaya exponents* of $\boldsymbol{c}$. It can be shown that for any balance $\boldsymbol{c}$ the number -1 is always one Kowalevskaya exponent with the associated eigenvector $\boldsymbol{wc}$ (see, for example, [2,9]). Moreover, if $(c_1, \dots, c_n)$ is a balance of (1), then so is $(\mu^{s_1}c_1, \dots, \mu^{s_n}c_n)$, where $\mu^{d-1} = 1$. Thus these $d-1$ balances have the same Kowalevskaya exponent and form an equivalence class. In this paper we shall call such classes the Kowalevskaya equivalence classes.

The study of integrability, the objective of this paper, has always been one of the main topics in ordinary differential equations. There is rich reference on this topic with various kinds of background, theory, and applications. Recently, investigation of polynomial and rational integrability of quasi-homogeneous polynomial systems has been caught much attention, see, for example, [1–4,6–8] and references therein. In particular, for instance, in [2–4], it was established a link between the Kowalevskaya exponents of the quasi-homogeneous polynomial differential systems and the degree of their quasi-homogeneous polynomial first integrals. For the sake of comparison with our results, we summarize the related known facts into the following theorem.

Theorem 1. (See [2–4]) *Consider an $(s_1, \dots, s_n)$-d type system of the form* (1). *For each balance $\boldsymbol{c}$, let $\rho_1, \dots, \rho_n$ be the Kowalevskaya exponents associated with $\boldsymbol{c}$. If system* (1) *has an $(s_1, \dots, s_n)$-m type polynomial first integral, then there exist non-negative integers $k_1, \dots, k_n$ satisfying $k_1 + \dots + k_n \leqslant m$ such that $k_1\rho_1 + \dots + k_n\rho_n = m/(d-1)$.*

Tsyhvintsev in [8] and Llibre and Zhang in [4] studied $(1, \dots, 1)$-2 type systems. They proved that the coefficient corresponding to the Kowalevskaya exponent -1 in Theorem 1 can be taken to be 0. In this paper, we show that this property also holds for systems of $(s_1, \dots, s_n)$-d type.

Theorem 2. *For each balance $\boldsymbol{c}$ of an $(s_1, \dots, s_n)$-d type polynomial system* (1)*, $d \geqslant 2$, let $\rho_1 = -1, \rho_2, \dots, \rho_n$ be the Kowalevskaya exponents associated with $\boldsymbol{c}$. If* (1) *has an $(s_1, \dots, s_n)$-m type polynomial first integral, then there exist non-negative integers $k_2, \dots, k_n$, satisfying $k_2 + \dots + k_n \leqslant m$ such that $k_2\rho_2 + \dots + k_n\rho_n = m/(d-1)$.*

When restricting the study to differential systems on $\mathbf{C}^2$, we also extend the discussion of [4] and obtain more systematic results. More precisely, in [4], Llibre and Zhang considered the planar system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{2}$$

where $P(x, y) \neq 0$, $Q(x, y) \neq 0$, and P and Q are co-prime in $\mathbf{C}[x, y]$. The following results are from [4].

Theorem 3. (See [4]) *Assume that the planar system* (2) *is quasi-homogeneous polynomial system of weight degree $d = 2$, $P(x, y) \neq 0$, $Q(x, y) \neq 0$, and P and Q are co-prime. The following statements hold.*

(i) *If system* (2) *has two independent balances with the Kowalevskaya exponents* $(-1, \rho_1)$ *and* $(-1, \rho_2)$*, then it has a quasi-homogeneous polynomial first integral of weight degree $m \in \mathbf{N}$ if and only if $\rho_1^{-1} + \rho_2^{-1} \leqslant 1$, $\frac{m}{\rho_i} \in \mathbf{N}$, for $i = 1, 2$.*
(ii) *If system* (2) *has a unique balance with the Kowalevskaya exponents* $(-1, \rho)$*, $\rho \neq 0$, then it has a quasi-homogeneous polynomial first integral of weight degree $m \in \mathbf{N}$ if and only if $\rho^{-1} \leqslant 1$, $\frac{m}{\rho} \in \mathbf{N}$.*
(iii) *If system* (2) *has no balance or has infinitely many balances, then it has no quasi-homogeneous polynomial first integrals.*

We note that the case (iii) in Theorem 3, however, never occurs under the assumption of the theorem. In fact, one can show that if P and Q are co-prime then there are always at least one and at most three balances with all the three possibilities (see Lemma 10). Some more careful consideration in all these three possible cases convinces one the following: the system has a polynomial first integral of degree m if and only if for every possible balance with the corresponding Kowalevskaya exponent $(-1, \rho)$, $m/\rho \in \mathbf{N}$.

In the next theorem, on the one hand we generalize the above observation from (s_1, s_2)-2 type systems to systems of (s_1, s_2)-d type, $d \geqslant 2$, whereas on the other hand we extend the previous discussion from polynomial first integrals to rational first integrals.

Theorem 4. *Let the planar quasi-homogeneous polynomial differential system* (2) *be* (s_1, s_2)-d *type, $d \geqslant 2$, where $P(x, y)$ and $Q(x, y)$ are co-prime in* $\mathbb{C}[x, y]$. *Then the system always has at least one and at most $d + 1$ balances, and it has a polynomial* (*resp. rational*) *first integral if and only if for every balance* $\mathbf{c}$ *of the system with the corresponding Kowalevskaya exponent* $(-1, \rho)$*,*

$$\rho \in \mathbb{Q}^+ \quad (\textit{resp. } \rho \in \mathbb{Q} \setminus \{0\}).$$

In this paper we shall mainly work in the $\mathbf{C}^n$ category although a parallel study can be carried over to quasi-homogeneous polynomial differential systems in $\mathbf{R}^2$. For instance, the above theorem has an immediate corollary about planar homogeneous polynomial differential systems of degree $d \geqslant 2$.

Corollary 5. *Let the planar homogeneous polynomial differential system* (2) *be* $(1, 1)$-d *type, $d \geqslant 2$, where $P(x, y)$ and $Q(x, y)$ are co-prime in* $\mathbb{R}[x, y]$. *Then the system always has at least one and at most $d + 1$ balances, and it has a polynomial* (*resp. rational*) *first integral if and only if for every balance* $\mathbf{c}$ *of the system with the corresponding Kowalevskaya exponent* $(-1, \rho)$*,*

$$\rho \in \mathbb{Q}^+ \quad (\textit{resp. } \rho \in \mathbb{Q} \setminus \{0\}).$$

The structure of the paper is as follows. In the next section, a series of lemmas and some useful facts are collected. Section 3 consists of a detailed proof of Theorems 2 and 4. As an application of the results, in Section 4, we present an illustrative example to show the non-existence of polynomial first integrals. The example has been studied in [5], where the Darbouxian invariants are investigated. In this paper, we give a very elementary proof by considering the relationship between the balances and the exponents.

2. Preliminaries

In this part of the paper, we present some lemmas which will be used in the proof of our theorems.

Lemma 6. *Let H be a polynomial of its variables with the decomposition form $H = H_m + H_{m+1} + \cdots + H_{m+l}$, where H_{m+i} is a quasi-homogeneous polynomial of $\mathbf{s}$-$(m+i)$ type, $i = 0, \ldots, l$. Then H is a polynomial first integral (resp. integrating factor) of quasi-homogeneous system (1) if and only if each quasi-homogeneous part H_{m+i} is its first integral (resp. integrating factor).*

One can check the validity of the lemma straightforwardly, or see, for example, [4]. For the proof of the following lemma, we refer the reader to [5].

Lemma 7. *Let $M \in \mathbb{C}_{n\times n}$ be a square matrix of order n with eigenvalues $\rho_1, \ldots, \rho_n$, $\chi \in \mathbb{C}$. Assume that $h(\boldsymbol{x})$ is a homogeneous polynomial of degree d. Then the following statements hold.*

(i) *If $h(\boldsymbol{x})$ satisfies the equation*

$$\left\langle \nabla_{\boldsymbol{x}} h(\boldsymbol{x}), M\boldsymbol{x} \right\rangle = \chi h(\boldsymbol{x}), \tag{3}$$

where $\nabla_{\boldsymbol{x}} := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, then there exist non-negative integers $k_1, \ldots, k_n$ such that

$$\sum_{i=1}^{n} \rho_i k_i = \chi, \qquad \sum_{i=1}^{n} k_i = d. \tag{4}$$

(ii) *If system (4) has a unique solution $(k_1, \ldots, k_n) = (0, \ldots, 0, k_{i_0}, 0, \ldots, 0)$ and M is diagonalizable, then $h(\boldsymbol{x})$ must have the form $h(\boldsymbol{x}) = c x_{i_0}^d$ for some $c \in \mathbb{C}$.*

The following lemma is basic and its proof can be found in many textbooks of calculus.

Lemma 8. *Assume that $F(x)$ and $G(x)$ are co-prime in $\mathbb{C}[x]$ and $\deg F < \deg G = g$, where $\deg F$ means the degree of F. Let a be the coefficient of term x^g in $G(x)$ and b that of term x^{g-1} in $F(x)$. Then the following two statements hold:*

(i) *If $x_1, x_2, \ldots, x_l$ are the roots of $G(x)$ with multiplicity $n_1, n_2, \ldots, n_l$, respectively, then*

$$\frac{F(x)}{G(x)} = \sum_{i=1}^{l} \sum_{j=1}^{n_i} \frac{t_{i,j}}{(x - x_i)^j}, \tag{5}$$

where $t_{i,n_i} \neq 0$ for $i = 1, \ldots, l$.

(ii) *If $G(x)$ has simple roots only, $x_1, \dots, x_g$, then*

$$\frac{F(x)}{G(x)} = \sum_{i=1}^{g} \frac{t_i}{x - x_i}, \tag{6}$$

where

$$t_i = \frac{F(x_i)}{G'(x_i)} \neq 0, \qquad \sum_{i=1}^{g} t_i = \frac{a}{b}. \tag{7}$$

Lemma 9. *Consider quasi-homogeneous polynomial differential system* (2) *of* (s_1, s_2)*-*d *type,* $d \geqslant 2$*, where* P *and* Q *are co-prime in* $\mathbb{C}[x, y]$*. Then there is a* 1–1 *correspondence between a balance* (c_1, c_2) *and an invariant curve* $l(x, y) = 0$ *of the system in the sense of the Kowalevskaya equivalence class, where* $l(x, y) = c_2^{s_1} x^{s_2} - c_1^{s_2} y^{s_1}$ *if* $c_1 c_2 \neq 0$*,* $l(x, y) = x$ *if* $c_1 = 0$ *and* $l(x, y) = y$ *if* $c_2 = 0$*.*

Proof. When $c_1 c_2 \neq 0$, we parameterize the curve $l(x, y) = 0$

$$\{(x, y)\colon\ l(x, y) = 0\} = \{(c_1 t^{s_1}, c_2 t^{s_2})\colon\ t \in \mathbb{C}\}$$

and compute the derivative of the curve along the vector field.

$$l_x P + l_y Q|_{l=0} = c_1^{s_2-1} c_2^{s_1-1} t^{s_1 s_2 + d - 1} \big(c_2 s_2 P(c_1, c_2) - c_1 s_1 Q(c_1, c_2)\big) \equiv 0.$$

Therefore the curve $l(x, y) = 0$ is invariant.

When $c_1 = 0$, then for $\forall y \in \mathbb{C}$, we have $P(0, y) = t^{d+s_1-1} P(0, c_2) \equiv 0$, where t satisfies $t^{s_2} c_2 = y$. Thus $x | P(x, y)$, i.e., the line $x = 0$ is invariant. A similar discussion can be given in the case $c_2 = 0$.

Conversely, if $l(x, y) = b^{s_1} x^{s_2} - a^{s_2} y^{s_1} = 0$, $ab \neq 0$, is invariant, then parameterize it and compute the derivative along the vector field. This yields

$$b s_2 P(a, b) - a s_1 Q(a, b) = 0.$$

Namely, $P(a, b) = t_0 a s_1$, $Q(a, b) = t_0 b s_2$. Notice that $t_0 \neq 0$, since otherwise, by the Hilbert Basis Theorem, $l(x, y) | (P, Q)$, which leads to a contradiction to the assumption. Hence one sees that $(\lambda^{s_1} a, \lambda^{s_2} b)$ is a balance, where $(d - 1)\lambda^{d-1} t_0 + 1 = 0$.

If $x = 0$ is invariant, then $x | P$. The co-prime assumption implies that $Q(0, y) \neq 0$ for any $y \neq 0$. Let $Q(0, y_0) = q_0$, where $y_0 \neq 0$ and $q_0 \neq 0$. Then it is straightforward to verify that $(0, t_0^{s_2} y_0)$ is a balance, where $(1 - d) t_0^{d-1} q_0 = s_2 y_0$.

The case that $y = 0$ is invariant can be argued in a similar way. □

As a particular case of Lemma 9, we see that if the system is homogeneous, i.e., $(s_1, s_2) = (1, 1)$, $d \geqslant 2$, then the 1–1 correspondence is established between balances and invariant lines of the system.

To prove Theorem 4 for a quasi-homogeneous (s_1, s_2)-d type system, $d \geqslant 2$, we can perform changes of coordinates and reduce the transformed system to a homogeneous $(1, 1)$-m type system. The degree m, however, depends on the original system and takes values $m \geqslant 0$. Thus, we have to establish some properties for $(1, 1)$-m type systems to include the cases $m = 0$ and $m = 1$.

Consider the following homogeneous polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{8}$$

where $P(x,y)$ and $Q(x,y)$ are co-prime with the form

$$\begin{aligned} P(x,y) &= a_{d,0}x^d + a_{d-1,1}x^{d-1}y + \cdots + a_{0,d}y^d, \\ Q(x,y) &= b_{d,0}x^d + b_{d-1,1}x^{d-1}y + \cdots + b_{0,d}y^d, \quad d \geqslant 0. \end{aligned} \tag{9}$$

Denote by

$$M(u) := Q(1,u) - uP(1,u), \qquad N(v) := P(v,1) - vQ(v,1). \tag{10}$$

To each line passing through the origin with the form $l(x,y) = bx - ay = 0$, we assign it a *direction* (a,b) and a *character* κ:

$$\kappa = \begin{cases} -\frac{M'(\frac{b}{a})}{P(1,\frac{b}{a})} & \text{if } a \neq 0, \\ -\frac{N'(0)}{Q(0,1)} & \text{if } a = 0. \end{cases} \tag{11}$$

Then the following statements hold.

Lemma 10. *Consider system* (8). *The following statements hold.*

(i) *if $M(u) \equiv 0$, then $d = 1$, the system has infinitely many invariant lines passing through the origin and has rational but not polynomial first integrals.*
(ii) *if $M(u) \not\equiv 0$, then it has at least* 1 *and at most $d+1$ complex invariant lines passing through the origin. In this case, the system has rational* (*resp. polynomial*) *first integrals if and only if for each complex invariant line its corresponding character $\kappa \in \mathbb{Q} \setminus \{0\}$* (*resp. $\kappa \in \mathbb{Q}^+$*).

Proof of Lemma 10. It is obvious that the line $x = 0$ is invariant if and only if $x|P$, or equivalently, if and only if $a_{0,d} = 0$. All the other invariant lines taking the form $l(x,y) = y - u_0x = 0$ is invariant if and only if u_0 is the root of $M(u) = 0$. Therefore if $M(u) \equiv 0$ then it is possible only when $d = 1$, $P = kx$ and $Q = ky$, where $k \in \mathbb{C} \setminus \{0\}$. In this case any line passing through the origin is invariant. Straightforward calculation yields a rational first integral of the system.

In the case $M(u) \not\equiv 0$, if $a_{0,d} \neq 0$, then $\deg M = d + 1 \geqslant 1$. Thus $M(u) = 0$ has at least 1 and at most $d+1$ mutually different roots. In other words, the system has at least 1 and at most $d+1$ invariant lines.

If $a_{0,d} = 0$, then $\deg M \leqslant d$. Thus $M(u) = 0$ has at least 0 and at most d mutually different roots. We note that in this case there is always an additional invariant line, $x = 0$. Therefore the system also has at least 1 and at most $d+1$ invariant lines.

Now to prove the necessary and sufficient conditions in the statement, we also divide the discussion into two cases, $a_{0,d} \neq 0$ and $a_{0,d} = 0$.

If $a_{0,d} \neq 0$, then $\deg M = d + 1 > \deg P(1,u)$, $P(1,u)$ and $M(u)$ are co-prime. We note that under the substitution $y = xu$ the system can be integrated explicitly, namely,

$$\int \frac{P(1,u)}{M(u)}\,\mathrm{d}u = \ln x.$$

By Lemma 8, we see that the system has a rational first integral if and only if: (i) the decomposition of the integrand in the left side of the above equation has the form

$$\frac{P(1,u)}{M(u)} = \sum_{i=1}^{d+1} \frac{t_i}{u - u_i},$$

where $t_i = \frac{P(1,u_i)}{M'(u_i)} \neq 0$, $\sum_{i=1}^{d+1} t_i = -1$; (ii) there exists a constant C such that $t_i/C \in \mathbb{Q} \setminus \{0\}$. Since $\sum_{i=1}^{d+1} t_i = -1$ therefore $C \in \mathbb{Q}$. It follows that $t_i \in \mathbb{Q}$. Thus by definition $\kappa_i = -t_i^{-1} \in$, and $\sum_{i=1}^{d+1} \kappa_i^{-1} = 1$. On the other hand, if all the $\kappa_i \in \mathbb{Q} \setminus \{0\}$, then $M(u)$ has only simple root, this is because for any root u_{i_0} of $M(u)$ with multiplicity $\geqslant 2$, we have $M'(u_{i_0}) = 0$, hence $\kappa = 0$. For $\deg M(u) = d+1$, we have exact $d+1$ non-zero characters. From the proof of the necessity, it is easy to check that $\sum_{i=1}^{d+1} \kappa_i^{-1} = 1$. Moreover, the system has a first integral of the form

$$H = \prod_{i=1}^{d+1} (y - u_i x)^{-\kappa_i^{-1}}.$$

Therefore, the system (8) has a rational integral if and only if all the $\kappa \in \mathbb{Q} \setminus \{0\}$. A similar discussion shows that, it has a polynomial first integral if and only if all the $\kappa \in \mathbb{Q}^+$.

The discussion in the case $a_{0,d} = 0$ can be given accordingly, and we omit the details here. □

3. Proof of theorems

Proof of Theorem 2. Let $\boldsymbol{c} = (c_1, \ldots, c_n)$ be a balance of (1). Without loss of generality, we assume that $c_1 \neq 0$. Denote by $\rho_1 = -1$ one of the Kowalevskaya exponents corresponding to $\boldsymbol{c}$. Assume that e_i, $i = 1, \ldots, n$, the unit coordinate frame of $\mathbb{C}^n$. Choose a non-singular matrix $T = (\boldsymbol{wc}, e_2, \ldots, e_n)$ such that

$$T^{-1} K(\boldsymbol{c}) T = \begin{pmatrix} -1 & * \\ 0 & M \end{pmatrix} \tag{12}$$

where $M = (m_{i,j})_{(n-1)\times(n-1)}$ with

$$m_{i,j} = -\frac{s_{i+1}c_{i+1}}{s_1 c_1} \frac{\partial P_1}{\partial x_{j+1}} + \frac{\partial P_{i+1}}{\partial x_{j+1}} + \delta_{ij} \frac{s_{i+1}}{d-1},$$

where δ_{ij} is the Kronecker symbol.

If H is the quasi-homogeneous polynomial first integral of type $\boldsymbol{s}$-m of system (1) then on the one hand it satisfies

$$\sum_{i=1}^{n} P_i(\boldsymbol{x}) \frac{\partial H(\boldsymbol{x})}{\partial x_i} = 0$$

while on the other hand it satisfies the generalized Euler identity

$$\sum_{i=1}^{n} s_i x_i \frac{\partial H(\boldsymbol{x})}{\partial x_i} = mH(\boldsymbol{x}).$$

Thus we obtain

$$\sum_{i=2}^{n} \bigl(s_i x_i P_1(\boldsymbol{x}) - s_1 x_1 P_i(\boldsymbol{x})\bigr) \frac{\partial H}{\partial x_i} = mH(\boldsymbol{x}) P_1(\boldsymbol{x}).$$

Let $x_1 = c_1$ and $x_i = y_i + c_i$, $i = 2, \ldots, n$, we have

$$\sum_{i=2}^{n} \bigl(s_i (y_i + c_i) \widehat{P_1}(\boldsymbol{y}) - s_1 c_1 \widehat{P_i}(\boldsymbol{y})\bigr) \frac{\partial \widehat{H}(\boldsymbol{y})}{\partial y_i} = m\widehat{H}(\boldsymbol{y}) \widehat{P_1}(\boldsymbol{y}), \tag{13}$$

where $\boldsymbol{y} = (y_2, \dots, y_n)$, $\widehat{H}(\boldsymbol{y}) = H(c_1, c_2 + y_2, \dots, c_n + y_n)$ and $\widehat{P_i}(\boldsymbol{y}) = P_i(c_1, c_2 + y_2, \dots, c_n + y_n)$, $i = 2, \dots, n$. Note that in general $\widehat{H}$ is not a quasi-homogeneous polynomial in $n-1$ variables. However we can decompose $\widehat{H}(\boldsymbol{y})$ and $\widehat{P_i}(\boldsymbol{y})$ into the sums of polynomials of the usual degrees $\widehat{H} = \sum_{i=0}^{\deg \widehat{H}} \widehat{H}_i$ and $\widehat{P}_j = \sum_{i=0}^{\deg \widehat{P}_j} \widehat{P}_{j,i}$, $j = 1, \dots, n$. Let i_0 be the smallest number such that $\widehat{H}_{i_0}$ does not vanish. Then by comparing the lowest terms in both sides of Eq. (13), we obtain $i_0 \geqslant 1$ and the following identity.

$$\left\langle \nabla_{\boldsymbol{y}} \widehat{H}_{i_0}(\boldsymbol{y}), M\boldsymbol{y} \right\rangle = \frac{m}{d-1} \widehat{H}_{i_0}(\boldsymbol{y}),$$

where M remains the same as in (12). Thus by Lemma 7, we prove the theorem. □

Proof of Theorem 4. Let quasi-homogeneous polynomial system (2) be (s_1, s_2)-d type with a general form

$$\begin{cases} \dot{x} = P(x, y) = \sum_i a_{k_1+is_2, l_1-is_1} x^{k_1+is_2} y^{l_1-is_1}, \\ \dot{y} = Q(x, y) = \sum_i b_{k_2+is_2, l_2-is_1} x^{k_2+is_2} y^{l_2-is_1}, \end{cases} \tag{14}$$

where $k_1 s_1 + l_1 s_2 = s_1 + d - 1$, $k_2 s_1 + l_2 s_2 = s_2 + d - 1$, and $k_1, k_2, l_1, l_2 \in \mathbf{N} \cup \{0\}$. It is straightforward to see that under the transformation

$$u = x^{s_2}, \qquad v = y^{s_1}, \tag{15}$$

system (14) is changed to the following system which generally is not polynomial any more.

$$\dot{u} = \widehat{P}(u, v), \qquad \dot{v} = \widehat{Q}(u, v), \tag{16}$$

where

$$\begin{cases} \widehat{P}(u, v) = s_2 u^{1-\frac{1}{s_2}} P(u^{\frac{1}{s_2}}, v^{\frac{1}{s_1}}) = s_2 u \sum_i a_{k_1+is_2, l_1-is_1} u^{\frac{k_1-1}{s_2}+i} v^{\frac{l_1}{s_1}-i}, \\ \widehat{Q}(u, v) = s_1 v^{1-\frac{1}{s_1}} Q(u^{\frac{1}{s_2}}, v^{\frac{1}{s_1}}) = s_1 v \sum_i b_{k_2+is_2, l_2-is_1} u^{\frac{k_2}{s_2}+i} v^{\frac{l_2-1}{s_1}-i}. \end{cases} \tag{17}$$

According to the form of the transformed system, we divide the discussion into two cases.

Type A. When the original system has the following form

$$\dot{x} = P(x, y) = a y^{s_1-1}, \qquad \dot{y} = Q(x, y) = b x^{s_2-1},$$

where $s_1 s_2 = s_1 + s_2 + d - 1$, then the transformed system takes the form

$$\begin{cases} \dot{u} = \widehat{P}(u, v) = s_2 a K(u, v), \\ \dot{v} = \widehat{Q}(u, v) = s_1 b K(u, v), \end{cases} \tag{18}$$

where $K(u, v) = u^{1-\frac{1}{s_2}} v^{1-\frac{1}{s_1}}$. The trajectories of (18) are conjugate to those of the following reduced (1, 1)-0 type system

$$\dot{u} = \widetilde{P}(u, v) = s_2 a, \qquad \dot{v} = \widetilde{Q}(u, v) = s_1 b. \tag{19}$$

Type B. If the original system has a form different than type A

$$\begin{cases} \dot{x} = P(x, y) = \sum_{i=-i_1}^{i_2} a_i x^{k+1+is_2} y^{l-is_1}, \\ \dot{y} = Q(x, y) = \sum_{j=-j_1}^{j_2} b_j x^{k+js_2} y^{l+1-js_1}, \end{cases} \tag{20}$$

where $i_1 = [\frac{k+1}{s_2}]$, $i_2 = [\frac{l}{s_1}]$, $j_1 = [\frac{k}{s_2}]$, $j_2 = [\frac{l+1}{s_1}]$ and $k, l \in \mathbb{N} \cup \{0\}$ satisfying $ks_1 + ls_2 = d - 1$, then the transformed system takes the form

$$\begin{cases} \dot{u} = \widehat{P}(u, v) = s_2 K(u, v) \sum_{i=-i_1}^{i_2} a_i u^{[\frac{k}{s_2}]+1+i} v^{[\frac{l}{s_1}]-i}, \\ \dot{v} = \widehat{Q}(u, v) = s_1 K(u, v) \sum_{j=-j_1}^{j_2} b_j u^{[\frac{k}{s_2}]+j} v^{[\frac{l}{s_1}]+1-j}, \end{cases} \tag{21}$$

where $K(u,v)=u^{\frac{k}{s_2}-[\frac{k}{s_2}]}v^{\frac{l}{s_1}-[\frac{l}{s_1}]}$. The trajectories of (21) are conjugate to those of the following reduced $(1,1)$-m type system

$$\begin{cases} \dot{u}=\widetilde{P}(u,v)=s_2\sum_{i=-i_1}^{i_2} a_i u^{[\frac{k}{s_2}]+1+i}v^{[\frac{l}{s_1}]-i}, \\ \dot{v}=\widetilde{Q}(u,v)=s_1\sum_{j=-j_1}^{j_2} b_j u^{[\frac{k}{s_2}]+j}v^{[\frac{l}{s_1}]+1-j}, \end{cases} \tag{22}$$

where $m=[\frac{k}{s_2}]+[\frac{l}{s_1}]+1\geqslant 1$.

It is easy to check that $\widetilde{M}(u)=\widetilde{Q}(1,u)-u\widetilde{P}(1,u)\equiv 0$ if and only if (14) has the form $\dot{x}=ax$, $\dot{y}=by$, and thus $\deg P=\deg Q=1$. In other words, if the original quasi-homogeneous system has the weight degree $d\geqslant 2$, then the system (22) falls in the case (ii) in Lemma 10. Consequently, it is not possible for the system to have infinitely many balances.

We note that the difference between type A and type B systems lies in the forms of the reduced homogeneous systems. In other words, type A systems can be reduced to constant vector fields, whereas type B systems can be reduced to homogeneous polynomial systems of degree m, $m\geqslant 1$.

We first point out that if P and Q are co-prime in $\mathbb{C}[x,y]$ then $\widetilde{P}$ and $\widetilde{Q}$ are co-prime too. This is clear since if $(au+bv)|(\widetilde{P},\widetilde{Q})$ and $ab\neq 0$, then $(ay^{s_1}+bx^{s_2})|(P,Q)$. Since $l-s_1[\frac{l}{s_1}]\geqslant 0$ and $k-s_2[\frac{k}{s_2}]\geqslant 0$, therefore for the case $ab=0$, if $u|(\widetilde{P},\widetilde{Q})$ then $x|(P,Q)$; and if $v|(\widetilde{P},\widetilde{Q})$ then $y|(P,Q)$.

Since it is obvious that the quasi-homogeneous system (14) has a rational (resp. polynomial) first integral if and only if the corresponding reduced homogeneous system (19/22) has a rational (resp. polynomial) first integral, whereas the latter has rational (resp. polynomial) first integral if and only if the corresponding character κ is a non-zero rational (resp. positive rational) number. Therefore, to prove the theorem, we show that between the Kowalevskaya exponents ρ of (14) and the characters κ of (19/22) the following relation holds.

$$\rho=\frac{s_1s_2}{d-1}\kappa. \tag{23}$$

We only consider type B systems, since the proof to type A systems can be given accordingly. For any balance (c_1,c_2) of (14), if $c_1c_2\neq 0$, then (14) has an invariant curve $c_2^{s_1}x^{s_2}-c_1^{s_2}y^{s_1}=0$ and (22) has an invariant line $c_2^{s_1}u-c_1^{s_2}v=0$, therefore by Lemma 9 we obtain a balance $(\tilde{c}_1,\tilde{c}_2)$ of the system (22). If $c_1c_2=0$, without loss of generality, we assume that $c_1=0$. Then $x|P(x,y)$. Therefore if the coefficient of the x^{k+1+is_2} in (20) is not zero, then $k+is_2\geqslant 0$, $[\frac{k}{s_2}]+i\geqslant 0$. This shows that $u|\widetilde{P}$, we also have a corresponding balance $(0,\tilde{c}_2)$.

It is clear that the above map is one-one.

To establish the relation (23), on the one hand, we calculate the trace of the matrix $\boldsymbol{DP}(\boldsymbol{c})$ in the definition of the Kowalevskaya exponents. Namely, we have

$$\rho-1=\frac{\partial P}{\partial x}(c_1,c_2)+\frac{\partial Q}{\partial y}(c_1,c_2)+\frac{s_1+s_2}{d-1},$$

where (c_1,c_2) is the balance. On the other hand, by the Euler formula we have

$$s_1x\frac{\partial P}{\partial x}+s_2y\frac{\partial P}{\partial y}=(s_1+d-1)P(x,y).$$

If $c_1\neq 0$, then we can eliminate $\frac{\partial P}{\partial x}(c_1,c_2)$ from the above expressions and obtain the following

$$\rho=-\frac{s_2c_2}{s_1c_1}\frac{\partial P}{\partial y}(c_1,c_2)+\frac{\partial Q}{\partial y}(c_1,c_2)+\frac{s_2}{d-1}.$$

Since under the substitution

$$u = x^{s_2}, \qquad v = y^{s_1},$$

we have

$$\widehat{P}(u,v) = s_2 u^{1-\frac{1}{s_2}} P\big(u^{\frac{1}{s_2}}, v^{\frac{1}{s_1}}\big), \qquad \widehat{Q}(u,v) = s_1 v^{1-\frac{1}{s_1}} Q\big(u^{\frac{1}{s_2}}, v^{\frac{1}{s_1}}\big).$$

It follows that

$$\widehat{P}(a,b) = -\frac{s_1 s_2}{d-1} a, \qquad \widehat{Q}(a,b) = -\frac{s_1 s_2}{d-1} b,$$

where $a = c_1^{s_2}$ and $b = c_2^{s_1}$. With some calculation, we have

$$\frac{\partial \widehat{P}}{\partial v}(a,b) = \frac{s_2 c_1^{s_2-1}}{s_1 c_2^{s_1-1}} \frac{\partial P}{\partial y}(c_1,c_2), \qquad \frac{\partial \widehat{Q}}{\partial v}(a,b) = \frac{s_2 - s_1 s_2}{d-1} + \frac{\partial Q}{\partial y}(c_1,c_2).$$

Therefore

$$\begin{aligned}
\rho &= -\frac{b}{a}\frac{\partial \widehat{P}}{\partial v}(a,b) + \frac{\partial \widehat{Q}}{\partial v}(a,b) + \frac{s_1 s_2}{d-1} \\
&= -\frac{b}{a}\frac{\partial \widehat{P}}{\partial v}(a,b) + \frac{\partial \widehat{Q}}{\partial v}(a,b) - \frac{\widehat{P}(a,b)}{a} \\
&= -\frac{s_1 s_2}{d-1}\frac{\widehat{M}'(\frac{b}{a})}{\widehat{P}(1,\frac{b}{a})} \\
&= -\frac{s_1 s_2}{d-1}\frac{\widetilde{M}'(\frac{b}{a})}{\widetilde{P}(1,\frac{b}{a})} \\
&= \frac{s_1 s_2}{d-1}\kappa,
\end{aligned}$$

where $\widehat{M}(x) = \widehat{Q}(1,x) - x\widehat{P}(1,x)$ and $\widetilde{M}(x) = \widetilde{Q}(1,x) - x\widetilde{P}(1,x)$.

If $c_1 = 0$ ($c_2 \neq 0$), then by considering $\widehat{N}(x)$, $\widetilde{N}(x)$, $\widehat{Q}(x,1)$, $\widetilde{Q}(x,1)$ instead of $\widehat{M}(x)$, $\widetilde{M}(x)$, $\widehat{P}(1,x)$, $\widetilde{P}(1,x)$, respectively, we also have

$$\rho = -\frac{s_1 s_2}{d-1}\frac{\widetilde{N}'(0)}{\widetilde{Q}(0,1)} = \frac{s_1 s_2}{d-1}\kappa.$$

The theorem is proved. □

4. Applications

In this part of the paper, we apply the above results to the following example to show the non-existence of polynomial first integrals. The example has been studied in [5], where the Darbouxian invariants are investigated. Below we give a very elementary proof by considering the relationship between the balances and the exponents.

Example 11. The system of differential equations

$$\dot{x}_i = x_i x_{i+1}, \quad i = 1, \ldots, n, \tag{24}$$

where $x_{n+1} = x_1$ and $n \geqslant 3$, has no polynomial first integrals.

Remark that if $n=2$, then the system has a non-trivial polynomial first integral $H=x_1-x_2$.

Proof. First of all, it is obvious that $(-1,\dots,-1)$ is the only balance with exponents $(\rho_1,\dots,\rho_{n-1},-1)$. Now assume that H is a polynomial first integral of the system. Denote by $\widetilde{H}(y_1,\dots,y_{n-1})=H(-1+y_1,\dots,-1+y_{n-1},-1)$, and assume that the degree of $\widetilde{H}$ is m. Then there are two subcases.

Case (i). n is odd. Denote by $\rho_n=-1$ and by $\rho_1,\dots,\rho_{n-1}$ all the other $n-1$ roots of $\rho^n=-1$. By Theorem 2, there exist non-negative integers $k_1,\dots,k_{n-1}$ such that

$$\sum_{i=1}^{n-1} k_i\rho_i=m \quad\text{and}\quad \sum_{i=1}^{n-1} k_i\leqslant m.$$

But this is not possible because

$$m=\left|\sum_{i=1}^{n-1} k_i\rho_i\right|<\sum_{i=1}^{n-1} k_i.$$

Case (ii). n is even. Denote by $\rho_1=1$, $\rho_n=-1$, $\rho_2,\dots,\rho_{n-1}$ all the other $n-2$ roots of $\rho^n=1$. By the similar inequality

$$m=\left|\sum_{i=1}^{n-1} k_i\rho_i\right|\leqslant\sum_{i=1}^{n-1} k_i:=\mu,$$

we have $k_i=0$ for $i\neq 1$ and $m=\mu$. That is to say, the lowest order of $\widetilde{H}$ is also m, i.e., $\widetilde{H}$ is homogeneous. Since the components of the exponent are mutually different and the matrix M as given in the proof of Theorem 2 is diagonalizable, therefore by Lemma 7, we have

$$\widetilde{H}=(l_1y_1+\cdots+l_{n-1}y_{n-1})^m,$$

where $l_1,\dots,l_{n-1}\in\mathbb{C}$. On the other hand, by the Euler identity, $\widetilde{H}$ satisfies the following relation

$$\sum_{i=1}^{n-2}(-1+y_i)(y_{i+1}-y_i)\frac{\partial\widetilde{H}}{\partial y_i}-y_1(-1+y_{n-1})\frac{\partial\widetilde{H}}{\partial y_{n-1}}=-m(-1+y_1)\widetilde{H}.$$

Therefore we have

$$\sum_{i=1}^{n-2}(y_i-1)(y_{i+1}-y_i)l_i-y_1(y_{n-1}-1)l_{n-1}=(1-y_1)\sum_{i=1}^{n-1}l_iy_i.$$

Straightforward comparison of the quadratic terms in both sides of the above equality yields $l_1=\cdots=l_{n-1}=0$. Thus the system has no polynomial first integrals. □

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